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# ASYMPTOTIC SYMMETRY FOR A CLASS OF QUASI-LINEAR PARABOLIC PROBLEMS

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Abstract. We study the symmetry properties of the weak positive solutions to a class of quasi-linear elliptic problems having a variational structure. On this basis, the asymptotic behaviour of global solutions of the corresponding parabolic equations is also investigated. In particular, if the domain is a ball, the elements of the $\omega$ limit set are nonnegative radially symmetric solutions of the stationary problem.

Contents



## 1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $1 < p < \infty$. The goal of this paper is to study the asymptotic symmetry properties for a class of global solutions of the following quasi-linear parabolic problem

$$(E)\qquad \begin{cases} u_t - \operatorname{div}(a(u)|\nabla u|^{p-2}\nabla u) + \frac{a'(u)}{p}|\nabla u|^p = f(u) & \text{in } (0,\infty)\times\Omega,\\ u(0,x) = u_0(x) & \text{in } \Omega,\\ u(t,x) = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}$$

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The adoption of the $p$-Laplacian operator inside the diffusion term arises in various applications where the standard linear heat operator $u_t - \Delta$ is replaced by a nonlinear diffusion with gradient dependent diffusivity. These models have been used in the theory of non-Newtonian filtration fluids, in turbulent flows in porous media and in glaciology (cf. [AE]). In the following we will assume that $a \in C^2_{\rm loc}(\mathbb{R})$ and there exists a positive constant $\eta$ such that $a(s) \geq \eta > 0$ for all $s \in \mathbb{R}^+$ and that $f$ is a locally lipschitz continuous in $[0, \infty)$, which satisfies some additional positivity conditions. The nontrivial (positive) stationary solutions of the above problem must be solutions of the following elliptic equation

$$(S) \qquad \begin{cases} -\mathrm{div}(a(u)|\nabla u|^{p-2}\nabla u) + \frac{a'(u)}{p}|\nabla u|^p = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This class of problems has been intensively studied with respect to existence, nonexistence and multiplicity via non-smooth critical point theory. For a quite recent survey paper, we refer the interested reader to [Sq] and to the references therein. Already in the investigation of the qualitative properties for the pure $p$-Laplacian case $a \equiv 1$, one has to face nontrivial difficulties mainly due to the lack of regularity of the solutions of problem ($S$). As known, the maximal regularity of bounded solutions in the interior of the domain is $C^{1,\alpha}(\Omega)$ (see [Di, Tol]). Also, since we are assuming the domain to be smooth, the $C^{1,\alpha}$ regularity assumption up to the boundary follows by [Lie]. In some sense, the problem is singular (for $1 < p < 2$) and degenerate (for $p > 2$) due to the different behaviour of the weight $|\nabla u|^{p-2}$.

**Definition 1.1.** *We denote by $\mathcal{S}_{x_1}$ the set of nontrivial weak $C^{1,\alpha}(\overline{\Omega})$ solutions $z$ of problem ($S$) which are symmetric and non-decreasing in the $x_1$-direction*[1]*. We denote by $\mathcal{R}$ the set of nontrivial weak $C^{1,\alpha}(\overline{\Omega})$ solutions $z$ of problem ($S$) which are radially symmetric and radially decreasing.*

The first result of the paper, regarding the stationary problem, is the following

**Theorem 1.2.** *Assume that $f$ is strictly positive in $(0, \infty)$ and $\Omega$ is strictly convex with respect to a direction, say $x_1$, and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then, a weak $C^{1,\alpha}(\overline{\Omega})$ solution $u$ of problem ($S$) belongs to $\mathcal{S}_{x_1}$. In addition, if $\Omega$ is a ball, then $u$ belongs to $\mathcal{R}$.*

Following also some ideas in [DS1], the main point in proving the above result is providing in this framework a suitable summability for the weight $|\nabla u|^{-1}$, allowing to prove that the set of critical points of $u$ has actually zero Lebesgue measure.

[1] As customary we consider the case of a domain which is symmetric with respect to the hyperplane $\{x_1 = 0\}$, and we mean that the solution $z$ is non-decreasing in the $x_1$-direction for $x_1 < 0$. While it is non-increasing for $x_1 > 0$.

**Definition 1.3.** *Given $u_0 \in W_0^{1,p}(\Omega)$ with $u_0 \geq 0$ a.e., we write $u_0 \in \mathcal{G}$, if there exists a function*

$$u \in C([0,\infty); W_0^{1,p}(\Omega, \mathbb{R}^+)), \quad u_t \in L^2([0,\infty); L^2(\Omega)), \qquad u(0) = u_0, \tag{1.1}$$

*solving the problem*

$$\int_0^T \int_\Omega u_t \varphi dx dt + \int_0^T \int_\Omega a(u)|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx dt + \int_0^T \int_\Omega \frac{a'(u)}{p}|\nabla u|^p \varphi dx dt = \int_0^T \int_\Omega f(u)\varphi dx dt, \quad \forall \varphi \in C_c^\infty(Q_T),$$

*for any $T > 0$, where $Q_T = \Omega \times [0,T]$ and satisfying the energy inequality*

$$\mathcal{E}(u(t)) + \int_s^t \int_\Omega |u_t(\tau)|^2 dx d\tau \leq \mathcal{E}(u(s)), \quad \textit{for all } t > s \geq 0, \tag{1.2}$$

*where the energy functional is defined as*

$$\mathcal{E}(u(t)) = \frac{1}{p}\int_\Omega a(u(t))|\nabla u(t)|^p dx - \int_\Omega F(u(t))dx, \qquad F(s) = \int_0^s f(\tau)d\tau.$$

As we learn from a (classical) work of Tsustumi [Ts, Theorems 1 to 4] regarding the pure $p$-Laplacian case (see also the works [Is, Zh]), the requirements (1.1) in Definition 1.3 are natural. In general, for the weak solutions of ($E$) to be globally defined, it is necessary that the initial datum $u_0$ is chosen sufficiently small. A similar consideration can be done for the size of the domain $\Omega$, sufficiently small domains yield global solutions, while large domains may yield to the appearance of blow-up phenomena. For well-posedness and Hölder regularity results for quasi-linear parabolic equation, we also refer the reader to the books [Di1, Li2]. Finally, concerning the energy inequality (1.2), of course smooth solutions of ($E$) will satisfy the energy identity (namely equality in (1.2) in place of the inequality). It is sufficient to multiply ($E$) by $u_t$ and, then, integrate in space and time. On the other hand (1.2) is enough for our purposes and it seems implicitly automatically satisfied by the Galerkin method yielding the existence and regularity of solutions, see e.g. [Ts, identity (3.8) and related weak convergences (3.9)-(3.13)].

The second result of the paper is the following

**Theorem 1.4.** *Assume that there exists a positive constant $\rho$ such that*

$$a'(s)s \geq 0, \quad \textit{for all } s \in \mathbb{R} \textit{ with } |s| \geq \rho, \tag{1.3}$$

*and that there exist two positive constants $C_1, C_2$ and $\sigma \in [1, p^* - 1)$ with $p > \frac{2n}{n+2}$, such that*

$$|f(s)| \leq C_1 + C_2|s|^\sigma, \qquad \textit{for all } s \in \mathbb{R} \tag{1.4}$$

*Then, the following facts hold.*

(a) *Assume that $f$ is strictly positive in $(0,\infty)$ and $\Omega$ is strictly convex with respect to a direction, say $x_1$, and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let $u_0 \in \mathcal{G}$ and let $u : [0,\infty) \times \Omega \to \mathbb{R}^+$ be the corresponding solution of (E). Then, for any diverging sequence $(\tau_j) \subset \mathbb{R}^+$ there exists a diverging sequence $(t_j) \subset \mathbb{R}^+$ with $t_j \in [\tau_j, \tau_j + 1]$ such that*

$$u(t_j) \to z \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } j \to \infty,$$

*where either $z = 0$ or $z \in \mathcal{S}_{x_1}$ (if $\Omega = B(0,R)$ with $R > 0$, then either $z = 0$ or $z \in \mathcal{R}$) provided that $z \in L^\infty(\Omega)$. In addition, for all $\mu_0 > 0$,*

$$\sup_{\mu\in[0,\mu_0]} \|u(t_j + \mu) - z\|_{L^q(\Omega)} \to 0 \quad \text{as } j \to \infty, \tag{1.5}$$

*for any $q \in [1, p^*)$.*

(b) *Let $R > 0$ and assume that $f \in C^1([0,\infty))$ with $f(0) = 0$ and*

$$0 < (p-1)f(s) < sf'(s), \qquad \text{for all } s > 0. \tag{1.6}$$

*Furthermore, assume that*

$$H'(s) \le 0 \quad \text{for } s > 0, \quad H(s) = (n-p)s - np\frac{\int_0^s f(\tau)d\tau}{f(s)}, \quad H(0) = 0. \tag{1.7}$$

*Let $u_0 \in \mathcal{G}$ and let $u : [0,\infty) \times B(0,R) \to \mathbb{R}^+$ be the corresponding solution of*

$$\begin{cases} u_t - \Delta_p u = f(u) & \text{in } (0,\infty) \times B(0,R), \\ u(0,x) = u_0(x) & \text{in } B(0,R), \\ u(t,x) = 0 & \text{in } (0,\infty) \times \partial B(0,R). \end{cases} \tag{1.8}$$

*Then, for any diverging sequence $(\tau_j) \subset \mathbb{R}^+$ there exists a diverging sequence $(t_j) \subset \mathbb{R}^+$ with $t_j \in [\tau_j, \tau_j + 1]$ such that*

$$u(t_j) \to z \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } j \to \infty,$$

*where either $z = 0$ or $z$ is the unique positive solution to the problem*

$$\begin{cases} -\Delta_p u = f(u) & \text{in } B(0,R), \\ u > 0 & \text{in } B(0,R), \\ u = 0 & \text{on } \partial B(0,R). \end{cases} \tag{1.9}$$

*In addition, the limit (1.5) holds.*

**Remark 1.5.** *The sign condition (1.3) is often assumed in the current literature on problem (S) (and in more general frameworks as well) in dealing with both existence and nonexistence results (see e.g. [CD, Sq, BBM]). We point out that it is, in general, necessary for the mere $W_0^{1,p}(\Omega)$ solutions to (S) to be bounded in $L^\infty(\Omega)$ (see [Fr]).*

Next, we consider a class of initial data corresponding to global solutions which enjoy some compactness over, say, the time interval $\{t > 1\}$.

**Definition 1.6.** *We write $u_0 \in \mathcal{A}$ if $u_0 \in \mathcal{G}$ and furthermore, the set*

$$K = \{u(t) : t > 1\},$$

*is relatively compact in $W_0^{1,p}(\Omega)$. For any initial datum $u_0 \in W_0^{1,p}(\Omega)$, the $\omega$-limit set of $u_0$ is defined as*

$$\omega(u_0) = \{z \in W_0^{1,p}(\Omega) : \text{there is } (t_j) \subset \mathbb{R}^+ \text{ with } u(t_j) \to z \text{ in } W_0^{1,p}(\Omega)\},$$

*where $u(t)$ is the solution of (E) corresponding to $u_0$.*

The third, and last, result of the paper is the following

**Theorem 1.7.** *Assume that $f$ is strictly positive in $(0, \infty)$ with the growth (1.4) and $\Omega$ is strictly convex with respect to a direction, say $x_1$, and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then, the following facts hold.*

(a) *For all $u_0 \in \mathcal{A}$, we have*

$$\omega(u_0) \cap L^\infty(\Omega) \subset \mathcal{S}_{x_1}.$$

*In particular, the $L^\infty$-bounded elements of the $\omega$-limit set to (E) with $\Omega = B(0, R)$ are zero or radially symmetric and decreasing solutions of problem (S).*

(b) *Assume that $f \in C^1([0, \infty))$ with $f(0) = 0$ satisfies assumptions (1.6) and (1.7). Then, for all $u_0 \in \mathcal{A}$, the $\omega$-limit set of problem (1.8) consists of either 0 or the unique positive solution to the problem (1.9).*

**Remark 1.8.** *Quite often, even in the fully nonlinear parabolic case, global solutions which are uniformly bounded in $L^\infty$ are considered (see e.g.* [Po, Section 3.1]*). In these cases, in our framework, the elements of the $\omega$-limit set are automatically bounded and, in turn, belong to $C^{1,\alpha}(\overline{\Omega})$. Concerning the $L^\infty$-global boundedness issue for a class of degenerate operators, such as the p-Laplacian case, we refer the reader to the work of Lieberman* [Li1]*, in particular* [Li1, Theorem 2.4]*, where he proves that*

$$\sup_{(t,x)\in[0,\infty)\times\Omega} |u(t, x)| < \infty,$$

*provided that suitable growth conditions hold on the parabolic operator as well as on the nonlinearity, which satisfy a typical super-linearity condition, reading as*

$$f(s)s \geq (a_0 + \alpha)F(s) - c_1, \quad F(s) \geq s^{2+\alpha} - c_0, \quad s \in \mathbb{R},$$

*for suitable positive constants $a_0, c_0, c_1$ and $\alpha$.*

**Remark 1.9.** *Assume that $\Omega$ is a star-shaped domain and consider the problem with the critical power nonlinearity*

$$\begin{cases} -\mathrm{div}(a(u)|\nabla u|^{p-2}\nabla u) + \frac{a'(u)}{p}|\nabla u|^p = u^{p^*-1} & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1.10}$$

*Assuming the sign condition*

$$a'(s) \geq 0, \quad \textit{for all } s \geq 0,$$

*it is known that problem (1.10) does not admit any solution (cf. [PS, DMS]). In turn, any uniformly bounded global solution to the problem*

$$\begin{cases} u_t - \mathrm{div}(a(u)|\nabla u|^{p-2}\nabla u) + \frac{a'(u)}{p}|\nabla u|^p = u^{p^*-1} & \textit{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x) & \textit{in } \Omega, \\ u(t,x) = 0 & \textit{in } (0,\infty)\times\partial\Omega \end{cases}$$

*must vanish along diverging sequences $(t_j) \subset \mathbb{R}^+$, $u(t_j) \to 0$ in $W_0^{1,p}(\Omega)$ as $j \to \infty$.*

**Remark 1.10.** *Theorems 1.2, 1.4, 1.7 are new already in the non-degenerate case $p = 2$ since of the presence of the coefficient $a(\cdot)$, in which case the solutions are expected to be very regular for $t > 0$.*

We do not investigate here conditions under which one can characterize a class of initial data which guarantee global solvability with the additional information of compactness of the trajectory into $W_0^{1,p}(\Omega)$. In the semi-linear case $p = 2$ with a power type nonlinearity $f(u) = |u|^{m-1}u$, $m > 1$, we refer to [CL, Qu, Qu1] for apriori estimates and smoothing properties in $C^1(\Omega)$ of the solutions for positive times. About the convergence to nontrivial solutions to the stationary problem along some suitable diverging time sequence $(t_j) \subset \mathbb{R}^+$, we also refer to [GW] for a detailed analysis of the sets of initial data $u_0 \in H_0^1(\Omega)$ yielding to vanishing and non-vanishing global solutions as well as initial data for which the solutions blow-up in finite time. In particular it is proved that the stabilization towards nontrivial equilibria is a borderline case, in the sense that the set of initial data corresponding to non-vanishing global solution is precisely the boundary of the (closed) set of data yielding global solutions. In conclusion, in general, at least four different type of behaviour may occur in these problems: blow up in finite time, global vanishing solution, global non-vanishing solution (converging to equilibria) and finally global solution blowing up in infinite time (see also [NST]). In our general framework, also due to the degenerate nature of the problem, this classification seems quite hard to prove, so we focus on the third case. In the $p$-Laplacian case $a \equiv 1$, we refer the reader to [Li1] for the study of apriori estimates and convergence to equilibria for global solutions. Our approach is based on the independent study of the symmetry properties of positive stationary solutions via a suitable weak comparison principle allowing to apply the Alexandrov-Serrin moving plane technique in symmetric domains (see also [DP, DS1, DS2] for similar results in the case

$a = 1$). Then, since the problem clearly admits a variational structure and the energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}(u(t)) = \frac{1}{p}\int_\Omega a(u(t))|\nabla u(t)|^p dx - \int_\Omega F(u(t))dx, \quad t > 0, \quad F(s) = \int_0^s f(\tau)d\tau,$$

is decreasing along a smooth solution $u(t)$, the global solutions have to approach stationary states along suitable diverging sequences $(t_j) \subset \mathbb{R}^+$. In pursuing this target we also make use of some nontrivial compactness result proved in [CD] in the study of the stationary problem. It is known that, in general, it is not possible to get the convergence result along the whole trajectory, namely as $t \to \infty$ (see [PoSi]) unless the nonlinearity $f$ is an analytic function (see [Je]).

For a general survey paper on the asymptotic symmetry of the solutions to general (not just those with a Lyapunov functional) nonlinear parabolic problems, we refer to the recent work of P. Poláčik [Po] where various different approaches to the study of the problem are discussed.

### Plan of the paper.

In Section 2 we study the regularity properties of the weak positive solutions to ($S$). In Section 3 we obtain some properties related to the asymptotic behaviour of solutions to the parabolic problem ($E$). Finally, in Section 4 we complete the proof of the main results of the paper.

### Notations.

(1) For $n \geq 1$, we denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^n$.
(2) $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) is the set of positive (resp. negative) real values.
(3) For $p > 1$ we denote by $L^p(\mathbb{R}^n)$ the space of measurable functions $u$ such that $\int_\Omega |u|^p dx < \infty$. The norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p(\Omega)}$.
(4) For $s \in \mathbb{N}$, we denote by $H^s(\Omega)$ the Sobolev space of functions $u$ in $L^2(\Omega)$ having generalized partial derivatives $\partial_i^k u$ in $L^2(\Omega)$ for all $i = 1, \dots, n$ and any $0 \leq k \leq s$.
(5) The norm $(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx)^{1/2}$ in $W_0^{1,p}(\Omega)$ is denoted by $\|\cdot\|_{W_0^{1,p}(\Omega)}$.
(6) We denote by $C_0^\infty(\Omega)$ the set of smooth compactly supported functions in $\Omega$.
(7) We denote by $B(x_0, R)$ a ball of center $x_0$ and radius $R$.
(8) We denote $D^2u$ the Hessian matrix of $u$ and $|D^2u|^2 \equiv \sum_{i=1}^n |\nabla u_i|^2$.
(9) We denote by $\mathcal{L}(E)$ the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

## 2. Symmetry for stationary solutions

We consider weak $C^{1,\alpha}(\overline{\Omega})$ solutions to ($S$). We recall that we shall assume that

(i) $f$ is locally lipschitz continuous in $[0, \infty)$;
(ii) For any given $\tau > 0$, there exists a positive constant $K$ such that $f(s) + Ks^q \geq 0$ for some $q \geq p - 1$ and for any $s \in [0, \tau]$. Observe that this implies $f(0) \geq 0$;

(iii) $a \in C^2_{\mathrm{loc}}(\mathbb{R})$ and there exists $\eta > 0$ such that $a(t) \geq \eta > 0$;

As pointed out in the introduction, if we assume that the solution is bounded, the $C^{1,\alpha}$ regularity up to the boundary follows by [Di, Tol, Lie]. Also hypothesis (iii) ensures the applicability of the Hopf boundary lemma (see [PS3, PSZ]).

2.1. **Gradients summability.** In weak form, our problem reads as

$$\int_\Omega a(u)|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \frac{1}{p}\int_\Omega a'(u)|\nabla u|^p \varphi dx = \int_\Omega f(u)\varphi dx, \quad \forall \varphi \in C^\infty_c(\Omega). \tag{2.1}$$

Define, as usual, the critical set $Z_u$ of $u$ by setting

$$Z_u = \left\{x \in \Omega : \nabla u(x) = 0\right\} \tag{2.2}$$

Note that the importance of critical set $Z_u$ is due to the fact that it is exactly the set where our operator is degenerate. By Hopf Lemma (cf. [PS3, PSZ]), it follows that

$$Z_u \cap \partial\Omega = \emptyset. \tag{2.3}$$

We want to point out that, by standard regularity results, $u \in C^2_{\mathrm{loc}}(\Omega \setminus Z_u)$. For functions $\varphi \in C^\infty_c(\Omega \setminus Z_u)$, let us consider the test function $\varphi_i = \partial_{x_i}\varphi$ and denote also $u_i = \partial_{x_i} u$, for all $i = 1, \dots, n$. With this choice in (2.1), integrating by part, we get

$$\begin{aligned}
&\int_\Omega a(u)|\nabla u|^{p-2}(\nabla u_i, \nabla\varphi) + (p-2)\int_\Omega a(u)|\nabla u|^{p-4}(\nabla u, \nabla u_i)(\nabla u, \nabla\varphi)dx \\
&\qquad + \int_\Omega a'(u)|\nabla u|^{p-2}(\nabla u, \nabla\varphi)u_i dx \\
&\qquad + \int_\Omega \frac{1}{p}a''(u)|\nabla u|^p u_i\varphi + \int_\Omega a'(u)|\nabla u|^{p-2}(\nabla u, \nabla u_i)\varphi \\
&\qquad - \int_\Omega f'(u)u_i\varphi = 0,
\end{aligned} \tag{2.4}$$

that is, in such a way, we have defined the linearized operator $L_u(u_i, \varphi)$ at a fixed solution $u$ of ($S$). Then we can write equation (2.4) as

$$L_u(u_i, \varphi) = 0, \qquad \forall \varphi \in C^\infty_c(\Omega \setminus Z_u). \tag{2.5}$$

In the following, we repeatedly use Young's inequality in this form

$$ab \leq \delta a^2 + C(\delta)b^2 \qquad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0.$$

We can now state and prove the following

**Proposition 2.1.** *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a solution to problem ($S$). Assume that $f$ is locally lipschitz continuous, $a \in C^2_{\mathrm{loc}}(\mathbb{R})$ and there exists a positive constant $\eta$ such that $a(s) \geq \eta > 0$ for all $s \in \mathbb{R}^+$. Assume that $\Omega$ is a bounded and smooth domain of $\mathbb{R}^n$. Then*

$$\int_{\Omega\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta}\, dx \leq \mathcal{C}, \tag{2.6}$$

*where $0 \leq \beta < 1$, $\gamma < n-2$ ($\gamma = 0$ if $n = 2$), $1 < p < \infty$ and the positive constant $\mathcal{C}$ does not depend on $y$. In particular, we have*

$$\int_{\Omega \setminus \{\nabla u = 0\}} \frac{|\nabla u|^{p-2-\beta} ||D^2 u||^2}{|y-x|^{\gamma}} \, dx \leq \tilde{\mathcal{C}}, \tag{2.7}$$

*for a positive constant $\tilde{\mathcal{C}}$ not depending on $y$.*

*Proof.* For all $\varepsilon > 0$, let us define the piecewise smooth function $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ by setting

$$G_\varepsilon(t) = \begin{cases} t & \text{if } |t| \geq 2\varepsilon, \\ 2t - 2\varepsilon & \text{if } \varepsilon \leq t \leq 2\varepsilon, \\ 2t + 2\varepsilon & \text{if } -2\varepsilon \leq t \leq -\varepsilon, \\ 0 & \text{if } |t| \leq \varepsilon. \end{cases} \tag{2.8}$$

Let us choose $E \subset\subset \Omega$ and a positive function $\psi \in C_c^\infty(\Omega)$, such that the support of $\psi$ is compactly contained in $\Omega$, $\psi \geq 0$ in $\Omega$ and $\psi \equiv 1$ in $E$. Let us set

$$\varphi_{\varepsilon,y}(x) = \frac{G_\varepsilon(u_i(x))}{|u_i(x)|^\beta} \frac{\psi(x)}{|y-x|^\gamma} \tag{2.9}$$

where $0 \leq \beta < 1$, $\gamma < n-2$ ($\gamma = 0$ for $n = 2$). Since $\varphi_{\varepsilon,y}$ vanishes in a neighborhood of each critical point, it follows that $\varphi_{\varepsilon,y} \in C_c^2(\Omega \setminus Z_u)$ and hence we can use it as a test

function in (2.4), getting the following result

$$
\begin{aligned}
&\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi|\nabla u_i|^2dx\\
&+\int_\Omega (p-2)\frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-4}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi(\nabla u,\nabla u_i)^2dx\\
&+\int_\Omega \frac{a'(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi u_i(\nabla u,\nabla u_i)dx\\
&+\int_\Omega \frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u_i,\nabla\psi)dx\\
&+\int_\Omega (p-2)\frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-4}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u,\nabla u_i)(\nabla u,\nabla\psi)dx\\
&+\int_\Omega \frac{a'(u)}{|y-x|^\gamma}|\nabla u|^{p-2}u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u,\nabla\psi)dx\\
&+\int_\Omega a(u)|\nabla u|^{p-2}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u_i,\nabla_x(\frac{1}{|y-x|^\gamma}))dx\\
&+\int_\Omega (p-2)a(u)|\nabla u|^{p-4}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u,\nabla u_i)(\nabla u,\nabla_x(\frac{1}{|y-x|^\gamma}))dx\\
&+\int_\Omega a'(u)|\nabla u|^{p-2}u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u,\nabla_x(\frac{1}{|y-x|^\gamma}))dx\\
&+\int_\Omega \frac{1}{p}a''(u)|\nabla u|^p u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx\\
&+\int_\Omega a'(u)|\nabla u|^{p-2}(\nabla u,\nabla u_i)\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx=\int_\Omega f'(u)u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx
\end{aligned}
$$

Let us denote each term of the previous equation in a useful way for the sequel, that is

$$
\begin{aligned}
A_1 &= \int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi|\nabla u_i|^2dx;\\
A_2 &= \int_\Omega (p-2)\frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-4}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi(\nabla u,\nabla u_i)^2dx;\\
A_3 &= \int_\Omega \frac{a'(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\Big(G'_\varepsilon(u_i)-\beta\frac{G_\varepsilon(u_i)}{u_i}\Big)\psi u_i(\nabla u,\nabla u_i)dx;\\
A_4 &= \int_\Omega \frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u_i,\nabla\psi)dx;\\
A_5 &= \int_\Omega (p-2)\frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-4}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u,\nabla u_i)(\nabla u,\nabla\psi)dx;\\
A_6 &= \int_\Omega \frac{a'(u)}{|y-x|^\gamma}|\nabla u|^{p-2}u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}(\nabla u,\nabla\psi)dx;\\
A_7 &= \int_\Omega a(u)|\nabla u|^{p-2}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u_i,\nabla_x(\frac{1}{|y-x|^\gamma}))dx;\\
A_8 &= \int_\Omega (p-2)a(u)|\nabla u|^{p-4}\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u,\nabla u_i)(\nabla u,\nabla_x(\frac{1}{|y-x|^\gamma}))dx;\\
A_9 &= \int_\Omega a'(u)|\nabla u|^{p-2}u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\psi(\nabla u,\nabla_x(\frac{1}{|y-x|^\gamma}))dx;\\
A_{10} &= \int_\Omega \frac{1}{p}a''(u)|\nabla u|^{p}u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx;\\
A_{11} &= \int_\Omega a'(u)|\nabla u|^{p-2}(\nabla u,\nabla u_i)\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx;\\
N &= \int_\Omega f'(u)u_i\frac{G_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi}{|y-x|^\gamma}dx.
\end{aligned}
\tag{2.10}
$$

Then we have rearranged the equation as

$$
\sum_{i=1}^{11} A_i = N \tag{2.11}
$$

Notice that, since $0\le\beta<1$, for all $t\in\mathbb{R}$ and $\varepsilon>0$ we have

$$
G'_\varepsilon(t)-\frac{\beta G_\varepsilon(t)}{t}\ge 0,\qquad \lim_{\varepsilon\to 0}\Big(G'_\varepsilon(t)-\frac{\beta G_\varepsilon(t)}{t}\Big)=1-\beta.
$$

From now on, we will denote

$$
\tilde G_\varepsilon(t)=G'_\varepsilon(t)-\beta\frac{G_\varepsilon(t)}{t},\qquad \text{for all } t\in\mathbb{R} \text{ and } \varepsilon>0.
$$

From equation (2.11) one has

$$A_1 + A_2 \leq \sum_{i=3}^{11} |A_i| + |N|.$$

We shall distinguish the proof into two cases.

**Case I: $\mathbf{p} \geq \mathbf{2}$**. This trivially implies $A_2 \geq 0$, and hence

$$A_1 \leq A_1 + A_2 \leq \sum_{i=3}^{11} |A_i| + |N|. \tag{2.12}$$

**Case II: $\mathbf{1} < \mathbf{p} < \mathbf{2}$**. By Schwarz inequality, of course, it follows

$$|\nabla u|^{p-4}(\nabla u, \nabla u_i)^2 \leq |\nabla u|^{p-2}|\nabla u_i|^2.$$

In turn, since $1 < p < 2$, this implies

$$(p-2)a(u)\frac{\tilde{G}_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi|\nabla u|^{p-4}(\nabla u, \nabla u_i)^2}{|y-x|^\gamma} \geq (p-2)a(u)\frac{\tilde{G}_\varepsilon(u_i)}{|u_i|^\beta}\frac{\psi|\nabla u|^{p-2}|\nabla u_i|^2}{|y-x|^\gamma},$$

so that $(p-2)A_1 \leq A_2$, yielding

$$A_1 \leq \frac{1}{p-1}(A_1 + A_2) \leq \frac{1}{p-1}\sum_{i=3}^{11} |A_i| + |N|. \tag{2.13}$$

In both cases, in view of (2.12) and (2.13), we want to estimates the terms in the sum

$$\sum_{i=3}^{11} |A_i| + |N|. \tag{2.14}$$

Let us start by estimating the terms $A_i$ in the sum (2.14). Concerning $A_3$, we have

$$\begin{aligned}
|A_3| &\leq \int_\Omega \frac{|a'(u)|}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\tilde{G}_\varepsilon(u_i)\psi|u_i||\nabla u||\nabla u_i|dx \\
&\leq C_3\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-1}}{|u_i|^\beta}\tilde{G}_\varepsilon(u_i)\psi|u_i||\nabla u_i|dx \\
&\leq C_3\left[\delta\int_\Omega \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{\tilde{G}_\varepsilon(u_i)}{|u_i|^\beta}\psi|\nabla u_i|^2dx + C_\delta\int_\Omega \frac{|\nabla u|^{p-1}}{|y-x|^\gamma}\psi\frac{\tilde{G}_\varepsilon(u_i)}{|u_i|^{\beta-2}}dx\right] \\
&\leq \frac{C_3\delta}{\eta}A_1 + K_3(\delta),
\end{aligned}$$

where we used that

$$|\nabla u|^{p-1}\psi\frac{\tilde{G}_\varepsilon(u_i)}{|u_i|^{\beta-2}} \leq C,$$

where $C$ is a positive constant independent of $\varepsilon$ and $C_3$ is a positive constant independent of $y$. Moreover recall that $0 \leq \beta < 1$ and that $u \in C^{1,\alpha}(\overline{\Omega})$. Also

$$|A_4| \leq \int_\Omega \frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}|\nabla u_i||\nabla\psi|dx \leq C_4,$$

where

$$\frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^{\beta-1}}\frac{|G_\varepsilon(u_i)|}{|u_i|}|\nabla u_i||\nabla\psi| \in L^\infty(\Omega),$$

since $|\nabla u_i|$ is bounded in a neighborhood of the boundary by Hopf Lemma, $\gamma-2<n$, $0\le\beta<1$ and the constant $C_4$ is independent of $y$. For the same reasons, we also have

$$\begin{aligned}|A_5| &\le \int_\Omega \frac{a(u)}{|y-x|^\gamma}|\nabla u|^{p-2}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}|\nabla u_i||\nabla\psi|dx \le C_5,\\ |A_6| &\le \int_\Omega \frac{|a'(u)|}{|y-x|^\gamma}|\nabla u|^{p-1}\frac{|G_\varepsilon(u_i)|}{|u_i|^{\beta-1}}|\nabla\psi|dx \le C_6,\end{aligned}$$

for some positive constants $C_5$ and $C_6$ independent of $y$. Furthermore, for a positive constant $C_7$ independent of $y$, we have

$$\begin{aligned}|A_7| &\le \int_\Omega a(u)|\nabla u|^{p-2}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\left|\nabla_x\frac{1}{|y-x|^\gamma}\right|dx\\ &\le C_7\int_\Omega a(u)|\nabla u|^{p-2}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\frac{1}{|y-x|^{\gamma+1}}dx\\ &\le C_7\delta\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{|G_\varepsilon(u_i)|}{|u_i|}|\nabla u_i|^2dx\\ &+C(\delta)\int_\Omega a(u)|\nabla u|^{p-1}\frac{|G_\varepsilon(u_i)|}{|u_i|}\frac{1}{|y-x|^{\gamma+2}}dx\\ &\le C_7\delta\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{|G_\varepsilon(u_i)|}{|u_i|}|\nabla u_i|^2dx + K_7(\delta)\end{aligned}$$

where we used Young's inequality, $\gamma-2<n$ and $0\le\beta<1$. In a similar fashion,

$$\begin{aligned}|A_8| &\le \int_\Omega |p-2|a(u)|\nabla u|^{p-2}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}\psi|\nabla u_i|\left|\nabla_x\frac{1}{|y-x|^\gamma}\right|dx\\ &\le C_8\delta\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{G_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx + K_8(\delta)\end{aligned}$$

as well as

$$|A_9| \le \int_\Omega |a'(u)||\nabla u|^{p-1}\frac{|G_\varepsilon(u_i)|}{|u_i|^{\beta-1}}\psi\left|\nabla_x\frac{1}{|y-x|^\gamma}\right|dx \le C_9.$$

for some positive constants $C_8, C_9$ independent of $y$. We get an upper bound for the last terms as well

$$|A_{10}| \le \frac{1}{p}\int_\Omega |a''(u)||\nabla u|^p\frac{|G_\varepsilon(u_i)|}{|u_i|^{\beta-1}}\frac{\psi}{|y-x|^\gamma}dx \le C_{10},$$

with $C_{10}$ independent of $y$ and where we have also used the fact that $a \in C^2_{\rm loc}(\mathbb{R})$. In the same way, it holds

$$
\begin{aligned}
|A_{11}| &\le \int_\Omega |a'(u)||\nabla u|^{p-1}\frac{|G_\varepsilon(u_i)|}{|u_i|^\beta}|\nabla u_i|\frac{\psi}{|y-x|^\gamma}dx\\
&\le C_{11}\delta\int_\Omega \frac{1}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\frac{G_\varepsilon(u_i)}{u_i}\psi|\nabla u_i|^2dx + C(\delta)\int_\Omega \frac{|\nabla u|^p}{|y-x|^\gamma}\frac{\psi}{|u_i|^{\beta-1}}\\
&\le \frac{C_{11}\delta}{\eta}\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\frac{G_\varepsilon(u_i)}{u_i}\psi|\nabla u_i|^2dx + K_{11}(\delta)
\end{aligned}
$$

and

$$
|N| \le \int_\Omega |f'(u)|\frac{|G_\varepsilon(u_i)|}{|u_i|^{\beta-1}}\frac{\psi}{|y-x|^\gamma}dx \le C_N,
$$

where the last inequality holds true since $f$ is locally lipschitz continuous and where $C_{11}$ and $C_N$ are constants independent of $y$. Then, by these estimates above and by equations (2.12), (2.13) and (2.14) we write

$$
A_1 \le \mathcal{D}\sum_{i=3}^{11}|A_i| + |N| \le \mathcal{S}\delta A_1 + \mathcal{M}\delta\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{G_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx + \mathcal{C}_\delta, \tag{2.15}
$$

where we have set

$$
\begin{aligned}
&\mathcal{D} = \max\Big\{1, \frac{1}{p-1}\Big\}, \quad \mathcal{S} = \mathcal{D}\frac{C_3}{\eta}, \quad \mathcal{M} = \mathcal{D}\max\Big\{C_7, C_8, \frac{C_{11}}{\eta}\Big\}\\
&\mathcal{C}_\delta = \max\big\{K_3(\delta), K_7(\delta), K_8(\delta), K_{11}(\delta), C_4, C_5, C_6, C_9, C_N\big\}.
\end{aligned}
$$

Then from equations (2.10) and (2.15) one has

$$
\begin{aligned}
&(1-\mathcal{S}\delta)\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\left(G'_\varepsilon(u_i) - \beta\frac{G_\varepsilon(u_i)}{u_i}\right)\psi|\nabla u_i|^2dx\\
&\le \mathcal{M}\delta\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\psi\frac{G_\varepsilon(u_i)}{u_i}|\nabla u_i|^2dx + \mathcal{C}_\delta,
\end{aligned}
$$

namely

$$
(1-\mathcal{S}\delta)\int_\Omega \frac{a(u)}{|y-x|^\gamma}\frac{|\nabla u|^{p-2}}{|u_i|^\beta}\left[G'_\varepsilon(u_i) - \left(\beta + \frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right)\frac{G_\varepsilon(u_i)}{u_i}\right]\psi|\nabla u_i|^2dx \le \mathcal{C}_\delta \tag{2.16}
$$

Let us choose $\delta > 0$ such that

$$
\begin{cases}
1-\mathcal{S}\delta > 0,\\
1-\left(\beta + \frac{\mathcal{M}\delta}{1-\mathcal{S}\delta}\right) > 0.
\end{cases} \tag{2.17}
$$

Therefore, since as $\varepsilon \to 0$

$$
\left[G'_\varepsilon(u_i) - \left(\beta + \frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right)\frac{G_\varepsilon(u_i)}{u_i}\right] \to \left(1-\beta-\frac{\mathcal{M}\delta}{(1-\mathcal{S}\delta)}\right) > 0, \quad \text{in } \{u_i \ne 0\},
$$

by Fatou's Lemma we get

$$\int_{\Omega\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta}\psi dx \le \mathcal{C}. \tag{2.18}$$

To prove (2.7) we choose $E \subset\subset \Omega$ such that

$$Z_u \cap (\Omega \setminus E) = \emptyset.$$

Since $u$ is $C^2$ in $\Omega \setminus E$, then we may reduce to prove that that

$$\int_{E\setminus\{u_i=0\}} \frac{|\nabla u|^{p-2}}{|y-x|^\gamma}\frac{|\nabla u_i|^2}{|u_i|^\beta} dx \le \mathcal{C}.$$

This, and hence the assertion, follows by considering (2.18) with a cut-off function as above with $\psi \in C_c^\infty(\Omega)$ positive, such that the support of $\psi$ is compactly contained in $\Omega$, $\psi \ge 0$ in $\Omega$ and $\psi \equiv 1$ in $E$. The proof is now complete. □

### 2.2. Summability of $|\nabla u|^{-1}$.

We have the following

**Theorem 2.2.** *Let $u$ be a solution of ($S$) and assume, furthermore, that $f(s) > 0$ for any $s > 0$. Then, there exists a positive constant $C$, independent of $y$, such that*

$$\int_\Omega \frac{1}{|\nabla u|^{(p-1)r}}\frac{1}{|x-y|^\gamma}\, dx \le C \tag{2.19}$$

*where $0 < r < 1$ and $\gamma < n-2$ for $n \ge 3$ ($\gamma = 0$ if $n = 2$).*

*In particular the critical set $Z_u$ has zero Lebesgue measure.*

*Proof.* Let $E$ be a set with $E \subset\subset \Omega$ and $(\Omega \setminus E) \cap Z_u = \emptyset$. Recall that $Z_u = \{\nabla u = 0\}$ and $Z_u \cap \partial\Omega = \emptyset$, in view of Hopf boundary lemma (see [PS3]). It is easy to see that, to prove the result, we may reduce to show that

$$\int_E \frac{1}{|\nabla u|^{(p-1)r}}\frac{1}{|x-y|^\gamma}\, dx \le C \tag{2.20}$$

To achieve this, let us consider the function

$$\Psi(x) = \Psi_{\varepsilon,y}(x) = \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}}\frac{1}{|x-y|^\gamma}\varphi, \tag{2.21}$$

where $0 < r < 1$ and $\gamma < n-2$ for $n \ge 3$ ($\gamma = 0$ if $n = 2$). We also assume that $\varphi$ is a positive $C_c^\infty(\Omega)$ cut-off function such that $\varphi \equiv 1$ in $E$. Using $\Psi$ as test function in ($S$),

since $f(u) \geq \sigma$ for some $\sigma > 0$ in the support of $\Psi$, we get

$$
\begin{aligned}
&\sigma \int_\Omega \Psi\, dx \leq \int_\Omega f(u)\Psi\, dx = \int_\Omega a(u)|\nabla u|^{p-2}(\nabla u, \nabla\Psi) + \frac{1}{p}a'(u)|\nabla u|^p \Psi\, dx \\
&\leq \int_\Omega a(u)|\nabla u|^{p-2}|(\nabla u, \nabla|\nabla u|)| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \frac{1}{|x-y|^\gamma}\varphi\, dx \\
&+ \int_\Omega a(u)|\nabla u|^{p-2}|(\nabla u, \nabla \frac{1}{|x-y|^\gamma})| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}}\varphi\, dx \\
&+ \int_\Omega a(u)|\nabla u|^{p-2}|(\nabla u, \nabla\varphi)| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma}\, dx \\
&+ \int_\Omega \frac{a'(u)}{p}|\nabla u|^p \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma}\varphi\, dx.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\int_\Omega \Psi\, dx \leq C\Big[\int_\Omega |\nabla u|^{p-1}|D^2 u| \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \frac{1}{|x-y|^\gamma}\varphi\, dx \\
&+ \int_\Omega \frac{|\nabla u|^{p-1}}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^{\gamma+1}}\varphi\, dx \\
&+ \int_\Omega \frac{|\nabla u|^{p-1}}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma}\, dx \\
&+ \int_\Omega \frac{|\nabla u|^{p}}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^\gamma}\, dx\Big].
\end{aligned}
$$

Then, denoting by $C_i$, suitable positive constants independent of $y$ and by $C_\delta$ a positive constant depending on $\delta$, we obtain

$$
\begin{aligned}
&\int_\Omega \Psi\, dx \leq C_1 \int_\Omega |\nabla u|^{p-1}|D^2 u| \cdot \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi\, dx \\
&+ C_2 \int_\Omega \frac{1}{|x-y|^{\gamma+1}}\, dx + C_3 \int_\Omega \frac{1}{|x-y|^\gamma}\, dx \\
&\leq C_1 \int_\Omega |\nabla u|^{p-1}|D^2 u| \cdot \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r+1}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi\, dx + C_4 \\
&\leq \delta C_5 \int_\Omega \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi\, dx \\
&+ C_\delta \int_\Omega |\nabla u|^{(p-2)-(p(r-1)+2-r)}|D^2 u|^2 \cdot \frac{1}{|x-y|^\gamma} \cdot \varphi\, dx + C_6 \leq \\
&\leq C_5 \delta \int_\Omega \Psi\, dx + C_\delta.
\end{aligned} \tag{2.22}
$$

Here we have we used that $u \in C^{1,\alpha}(\Omega)$, $\gamma < n-2$ and we have exploited the regularity result of Proposition 2.1. Then, by (2.22), fixing $\delta$ sufficiently small, such that $1 - C_5\delta > 0$,

one concludes

$$\int_{\Omega} \frac{1}{(|\nabla u| + \varepsilon)^{(p-1)r}} \frac{1}{|x-y|^{\gamma}} \varphi \, dx \leq K, \tag{2.23}$$

for some positive constant $K$ independent of $y$. Taking the limit for $\varepsilon$ going to zero, the assertion immediately follows by Fatou's Lemma. □

Proposition 2.2 provides in fact the right summability of the weight $\rho(x) = |\nabla u(x)|^{p-2}$ in order to obtain a weighted Poincaré inequality. We refer the readers to [DS1, Section 3] for further details. For the sake of selfcontainedness, we recall here the statement

**Theorem 2.3.** *If $u \in C^{1,\alpha}(\overline{\Omega})$ is a solution of $(S)$ with $f(s) > 0$ for $s > 0$, $p \geq 2$, then*

$$\|v\|_{L^q(\Omega)} \leq C_p(|\Omega|)\|\nabla v\|_{L^q(\Omega,\rho)}, \qquad \text{for every } v \in H^{1,q}_{0,\rho}(\Omega), \tag{2.24}$$

*where $\rho \equiv |\nabla u|^{p-2}$, $C_P(|\Omega|) \to 0$ if $|\Omega| \to 0$. In particular (2.24) holds for every function $v \in H^{1,2}_{0,\rho}(\Omega)$. Moreover if $p \geq 2$, $q \geq 2$ and $v \in W^{1,q}_0(\Omega)$, the same conclusion holds. In fact, being $u \in C^{1,\alpha}(\overline{\Omega})$, and $p \geq 2$, $\rho = |Du|^{p-2}$ is bounded, so that $W^{1,q}_0(\Omega) \hookrightarrow H^{1,q}_{0,\rho}(\Omega)$.*

Recall that, if $\rho \in L^1(\Omega)$, $1 \leq q < \infty$, the space $H^{1,q}_{\rho}(\Omega)$ is defined as the completion of $C^1(\overline{\Omega})$ (or $C^\infty(\overline{\Omega})$) under the norm

$$\|v\|_{H^{1,q}_\rho} = \|v\|_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega,\rho)} \tag{2.25}$$

where

$$\|\nabla v\|^q_{L^q(\Omega,\rho)} = \int_\Omega |\nabla v|^q \rho \, dx.$$

We also recall that $H^{1,q}_{0,\rho}$ may be equivalently defined as the space of functions having distributional derivatives represented by a function for which the norm defined in (2.25) is bounded. These two definitions are equivalent if the domain has piecewise regular boundary (as we are indeed assuming).

2.3. **Comparison principles.** We now have the following

**Proposition 2.4.** *Let $\tilde{\Omega}$ be a bounded smooth domain such that $\tilde{\Omega} \subseteq \Omega$. Assume that $u, v$ are solutions to the problem $(S)$ and assume that $u \leq v$ on $\partial\tilde{\Omega}$. Then there exists a positive constant $\theta$, depending both on $u$ and $f$, such that, assuming*

$$\mathcal{L}(\tilde{\Omega}) \leq \theta$$

*then it holds*

$$u \leq v \qquad \text{in } \tilde{\Omega}.$$

*Proof.* We start proving the result when $p > 2$. Let us recall the weak formulations

$$\int_\Omega a(u)|\nabla u|^{p-2}(\nabla u\,,\, \nabla\varphi) \;+\; \frac{a'(u)}{p}|\nabla u|^p \varphi \, dx = \int_\Omega f(u)\varphi \, dx, \tag{2.26}$$

$$\int_\Omega a(v)|\nabla v|^{p-2}(\nabla v\,,\, \nabla\varphi) \;+\; \frac{a'(v)}{p}|\nabla v|^p \varphi \, dx = \int_\Omega f(v)\varphi \, dx. \tag{2.27}$$

Then we assume by contradiction that the assertion is false, and consider

$$(u-v)^+ = \max\{u-v, 0\},$$

that, consequently, is not identically equal to zero. Let us also set $\Omega^+ \equiv \mathrm{supp}(u-v)^+ \cap \tilde{\Omega}$. Since by assumption $u \le v$ on $\partial\tilde{\Omega}$, it follows that $(u-v)^+ \in W_0^{1,p}(\tilde{\Omega})$. We can therefore choose it as admissible test function in (2.26) and (2.27). Whence, subtracting the two, we get

$$\begin{aligned}
&\int_{\Omega^+} a(u)|\nabla u|^{p-2}(\nabla u\,,\,\nabla(u-v)) - a(v)|\nabla v|^{p-2}(\nabla v\,,\,\nabla(u-v)) + \\
&+\int_{\Omega^+} \frac{a'(u)}{p}|\nabla u|^p(u-v)\,dx \,-\, \frac{a'(v)}{p}|\nabla v|^p(u-v)\,dx = \\
&= \int_{\Omega^+} (f(u)-f(v))(u-v)\,dx.
\end{aligned} \tag{2.28}$$

We can rewrite as follows

$$\begin{aligned}
&\int_{\Omega^+} a(u)((|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\,,\,\nabla(u-v)))\,dx \\
&+\int_{\Omega^+} (a(u)-a(v))|\nabla v|^{p-2}(\nabla v, \nabla(u-v))dx \\
&+\int_{\Omega^+} \frac{1}{p}(a'(u)-a'(v))|\nabla u|^p(u-v)\,dx \\
&+\int_{\Omega^+} \frac{a'(v)}{p}(|\nabla u|^p - |\nabla v|^p)(u-v)\,dx \\
&=\int_{\Omega^+} (f(u)-f(v))(u-v)\,dx.
\end{aligned} \tag{2.29}$$

First of all, since $a(u) \ge \eta > 0$, and using the fact that

$$\big(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi-\xi'\big) \ge c(|\xi|+|\xi'|)^{p-2}|\xi-\xi'|^2$$

for all $\xi, \xi' \in \mathbb{R}^n$, it follows that

$$c\eta\int_{\Omega^+} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \le \int_{\Omega^+} a(u)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u-v))\,dx \tag{2.30}$$

so that

$$\begin{aligned}
&\int_{\Omega^+} (|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \le C\int_{\Omega^+} |a(u)-a(v)||\nabla v|^{p-1}|\nabla(u-v)|dx + \\
&+C\int_{\Omega^+} |a'(u)-a'(v)||\nabla u|^p|u-v|\,dx \,+C\int_{\Omega^+} |a'(v)||\nabla u|^p - |\nabla v|^p||u-v|\,dx + \\
&+\int_{\Omega^+} |\frac{f(u)-f(v)}{u-v}||u-v|^2\,dx
\end{aligned} \tag{2.31}$$

Let us now evaluate the terms on right of the above inequality. By the smoothness of $a$, the $C^{1,\alpha}$ regularity of $u$, and exploiting Young inequality we get

$$
\begin{aligned}
&\int_{\Omega^+} |a(u)-a(v)||\nabla v|^{p-1}|\nabla(u-v)|dx \le C\int_{\Omega^+}|u-v||\nabla v|^{\frac{p-2}{2}}|\nabla(u-v)|dx \le \\
&\le C_\delta \int_{\Omega^+}(u-v)^2\,dx + \delta\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \le \\
&\le (C_\delta C_p(|\Omega^+|)+\delta)\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx.
\end{aligned}
\tag{2.32}
$$

Here $C_\delta$ is a constant depending on $\delta$, and $C_p(|\Omega^+|)$ is the Poincaré constant given by Theorem 2.3. Note in particular that, since $p>2$, we have $|\nabla u|^{p-2}\le(|\nabla u|+|\nabla v|)^{p-2}$. It is of course very important the fact that the constant $C_p(|\Omega^+|)$ goes to zero, provided that the Lebesgue measure of $\Omega^+$ goes to 0. Also we note that, by the $C^{1,\alpha}$ regularity of $u$, and exploiting the fact that $a'$ is Lipschitz continuous, we get

$$
\begin{aligned}
\int_{\Omega^+}|a'(u)-a'(v)||\nabla u|^p|u-v|\,dx &\le C\int_{\Omega^+}(u-v)^2\,dx \\
&\le C\,C_P(|\Omega^+|)\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx.
\end{aligned}
$$

Also, by convexity, we have

$$
\begin{aligned}
&\int_{\Omega^+}|a'(v)||\nabla u|^p-|\nabla v|^p||u-v|\,dx \\
&\le C\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{\frac{p-2}{2}}|\nabla(u-v)||u-v|\,dx \\
&\le \delta\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx + C_\delta\int_{\Omega^+}|u-v|^2\,dx \\
&\le \delta\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \\
&+ C_\delta C_P(|\Omega^+|)\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \\
&\le (\delta + C_\delta C_P(|\Omega^+|))\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx
\end{aligned}
\tag{2.33}
$$

Finally, by the Lipschitz continuity of $f$, it follows

$$
\begin{aligned}
\int_{\Omega^+}|\frac{f(u)-f(v)}{u-v}||u-v|\,dx &\le C\int_{\Omega^+}|u-v|^2\,dx \\
&\le C\,C_P(|\Omega^+|)\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx
\end{aligned}
$$

Concluding, exploiting the above estimates, we get

$$
\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx \le (\delta + C_\delta\,C_P(|\Omega^+|))\int_{\Omega^+}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2\,dx
$$

which gives a contradiction for $(\delta + C_\delta\, C_P(|\Omega^+|)) < 1$. Therefore, if we consider $\delta$ small fixed, say $\delta = \frac{1}{4}$, it then follows that also $C_\delta$ is fixed. Now, since $\mathcal{L}(\tilde{\Omega}) \leq \theta$ by assumption, it follows that if $\theta$ is sufficiently small, then we may assume that $C_P(|\Omega^+|)$ is also small, and that $C_\delta\, C_P(|\Omega^+|)) < \frac{1}{4}$. Consequently, it follows $(\delta + C_\delta\, C_P(|\Omega^+|)) < \frac{1}{2} < 1$, that leads to the above contradiction, and shows that actually $(u-v)^+ = 0$ and the thesis. The proof in the case $1 < p \leq 2$ in completely analogous, but is based on the classical Poincaré inequality. We give some details for the reader's convenience. Exactly as above we get (2.31). This , for $1 < p \leq 2$, considering the fact that the term $(|\nabla u| + |\nabla v|)^{p-2}$ is bounded below by the fact that $p - 2 \leq 0$ and $|\nabla u|\,,\, |\nabla v| \in L^\infty(\overline{\Omega})$, gives

$$
\begin{aligned}
&\int_{\Omega^+} |\nabla(u-v)|^2\, dx \leq C \int_{\Omega^+} |a(u) - a(v)||\nabla v|^{p-1}|\nabla(u-v)| dx + \\
&+ C \int_{\Omega^+} |a'(u) - a'(v)||\nabla u|^p |u-v|\, dx + C \int_{\Omega^+} |a'(v)|||\nabla u|^p - |\nabla v|^p||u-v|\, dx + \\
&+ \int_{\Omega^+} |\frac{(f(u) - f(v))}{(u-v)}| \cdot ||u-v|\, dx \leq \\
&C \int_{\Omega^+} |u-v||\nabla(u-v)| dx + C \int_{\Omega^+} |u-v|^2\, dx \ \leq \\
&\delta \int_{\Omega^+} |\nabla(u-v)|^2 dx + C_\delta \int_{\Omega^+} |u-v|^2\, dx \ \leq \\
&\delta \int_{\Omega^+} |\nabla(u-v)|^2 dx + C_\delta C_P(|\Omega^+|) \int_{\Omega^+} |\nabla(u-v)|^2\, dx \ \leq \\
&(\delta + C_\delta C_P(|\Omega^+|)) \int_{\Omega^+} |\nabla(u-v)|^2\, dx
\end{aligned}
\tag{2.34}
$$

For $\theta$ sufficiently small arguing as above we can assume $(\delta + C_\delta C_P(|\Omega^+|)) < 1$ which gives $(u-v)^+ = 0$ and the thesis. □

2.4. **The moving plane method.** Let us consider a direction, say $x_1$, for example. As customary we set

$$T_\lambda = \big\{x \in \mathbb{R}^n : x_1 = \lambda\big\}.$$

Given $x \in \mathbb{R}^n$, we define

$$x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n), \quad u_\lambda(x) = u(x_\lambda),$$

$$\Omega_\lambda = \big\{x \in \Omega : x_1 < \lambda\big\},$$

Set

$$\tilde{a} := \inf_{x \in \Omega} x_1.$$

Let $\Lambda$ be the set of those $\lambda > \tilde{a}$ such that for each $\mu < \lambda$ none of the conditions (i) and (ii) occurs, where

(i) The reflection of $(\Omega_\lambda)$ w.r.t. $T_\lambda$ becomes internally tangent to $\partial\Omega$ .
(ii) $T_\lambda$ is orthogonal to $\partial\Omega$.

We have the following

**Proposition 2.5.** *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a solution to the problem (S). Then, for any $\tilde{a} \leq \lambda \leq \Lambda$, we have*

$$u(x) \leq u_\lambda(x), \qquad \forall x \in \Omega_\lambda. \tag{2.35}$$

*Moreover, for any $\lambda$ with $\tilde{a} < \lambda < \Lambda$ we have*

$$u(x) < u_\lambda(x), \qquad \forall x \in \Omega_\lambda \setminus Z_{u,\lambda} \tag{2.36}$$

*where $Z_{u,\lambda} \equiv \{x \in \Omega_\lambda \,:\, \nabla u(x) = \nabla u_\lambda(x) = 0\}$. Finally*

$$\frac{\partial u}{\partial x_1}(x) \geq 0, \qquad \forall x \in \Omega_\Lambda. \tag{2.37}$$

*Proof.* For $\tilde{a} < \lambda < \Lambda$ and $\lambda$ sufficiently close to $\tilde{a}$, we assume that $\mathcal{L}(\Omega_\lambda)$ is as small as we like. We assume in particular that we can exploit the weak maximum principle in small domains (see Proposition 2.4) in $\Omega_\lambda$. Consequently, since we know that

$$u - u_\lambda \leq 0, \qquad \text{on } \partial\Omega_\lambda \tag{2.38}$$

by construction, by Proposition 2.4 it follows

$$u - u_\lambda \leq 0 \quad \text{in} \quad \Omega_\lambda.$$

We define

$$\Lambda_0 = \{\lambda > \tilde{a} : u \leq u_t, \text{ for all } t \in (\tilde{a}, \lambda]\} \tag{2.39}$$

and

$$\lambda_0 = \sup\, \Lambda_0. \tag{2.40}$$

Note that by continuity, we have $u \leq u_{\lambda_0}$. We have to show that actually $\lambda_0 = \Lambda$. Assume that by contradiction $\lambda_0 < \Lambda$ and argue as follows. Let $A$ be an open set such that $Z_u \cap \Omega_{\lambda_0} \subset A \subset \Omega_{\lambda_0}$. Note that since $|Z_u| = 0$ (see Theorem 2.2), we can choice $A$ as small as we like. Note now that by a strong comparison principle [PS3] we get

$$u < u_{\lambda_0} \qquad \text{or} \qquad u \equiv u_{\lambda_0}$$

in any connected component of $\Omega_{\lambda_0} \setminus Z_u$.
It follows now that

the case $u \equiv u_{\lambda_0}$ in some connected component $\mathcal{C}$ of $\Omega_{\lambda_0} \setminus Z_u$ is not possible.

The proof of this is completely analogous to the one given in [DP] once we have Proposition 2.4. Consider now a compact set $K$ in $\Omega_{\lambda_0}$ such that $|\Omega_{\lambda_0} \setminus K|$ is sufficiently small so that Proposition 2.4 works. By what we proved before, $u_{\lambda_0} - u$ is positive in $K \setminus A$ which is compact, therefore by continuity we find $\epsilon > 0$ such that, $\lambda_0 + \epsilon < \Lambda$ and for $\lambda < \lambda_0 + \epsilon$ we have that $|\Omega_\lambda \setminus (K \setminus A)|$ is still sufficiently small as before and $u_\lambda - u > 0$ in $K \setminus A$. In particular $u_\lambda - u > 0$ on $\partial(K \setminus A)$. Consequently $u \leq u_\lambda$ on $\partial(\Omega_\lambda \setminus (K \setminus A))$. By Proposition 2.4 it follows $u \leq u_\lambda$ in $\Omega_\lambda \setminus (K \setminus A)$ and consequently in $\Omega_\lambda$, which contradicts

the assumption $\lambda_0 < \Lambda$. Therefore $\lambda_0 \equiv \Lambda$ and the thesis is proved. The proof of (2.36) follows by the strong comparison theorem exploited as above. Finally (2.37) follow by the monotonicity of the solution that is implicitly in the above arguments. $\square$

## 3. Properties of the parabolic flow

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, and let $a : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that there exists positive constants $C, \nu$ and $\rho$ such that

$$\eta \le a(s) \le C, \ \ |a'(s)| \le C \quad \text{for all } s \in \mathbb{R}, \tag{3.1}$$

$$a'(s)s \ge 0, \quad \text{for all } s \in \mathbb{R} \text{ with } |s| \ge \rho. \tag{3.2}$$

As stated in the introduction, along any given global solution $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ of problem ($E$), and setting

$$F(s) = \int_0^s f(\tau)d\tau, \quad s \in \mathbb{R},$$

we also consider the energy functional $\mathcal{E}$ defined by

$$\mathcal{E}(u(t)) = \frac{1}{p}\int_\Omega a(u(t))|\nabla u(t)|^p dx - \int_\Omega F(u(t))dx,$$

and the related energy inequality (1.2). In particular, the energy functional $\mathcal{E}$ is non-increasing along solutions. Moreover, by the regularity we assumed on the global solutions, we have

$$\sup_{t>0} \|u(t)\|_{W_0^{1,p}(\Omega)} < \infty, \tag{3.3}$$

and

$$\int_0^\infty \int_\Omega |u_t(\tau)|^2 dx d\tau < \infty. \tag{3.4}$$

Next we state a quite useful result.

**Lemma 3.1.** *For all fixed $\mu_0 > 0$, it holds*

$$\lim_{t\to\infty} \sup_{\mu\in[0,\mu_0]} \|u(t) - u(t+\mu)\|_{L^q(\Omega)} = 0, \quad \textit{for all } q \in [1, p^*).$$

*If in addition the trajectory $\{u(t) : t > 1\}$ is relatively compact in $W_0^{1,p}(\Omega)$, we have*

$$\lim_{t\to\infty} \sup_{\mu\in[0,\mu_0]} \|u(t) - u(t+\mu)\|_{W_0^{1,p}(\Omega)} = 0,$$

*for all fixed $\mu_0 > 0$.*

*Proof.* Let us first prove that, for all $\mu_0 > 0$, it holds

$$\lim_{t\to\infty} \sup_{\mu\in[0,\mu_0]} \|u(t) - u(t+\mu)\|_{L^1(\Omega)} = 0. \tag{3.5}$$

Given $\mu > 0$, for all $t > 0$ and $\mu \in [0, \mu_0]$, from the energy inequality (1.2), we have

$$\begin{aligned}
\int_\Omega |u(t) - u(t+\mu)| dx &= \int_\Omega \Big| \int_t^{t+\mu} u_t(\tau) d\tau \Big| dx \le \int_t^{t+\mu} \int_\Omega |u_t(\tau)| d\tau dx \\
&\le \sqrt{\mu \mathcal{L}^n(\Omega)} \Big( \int_t^{t+\mu} \int_\Omega |u_t(\tau)|^2 d\tau dx \Big)^{1/2} \\
&\le \sqrt{\mu \mathcal{L}^n(\Omega)} (\mathcal{E}(u(t)) - \mathcal{E}(u(t+\mu)))^{1/2} \\
&\le \sqrt{\mu_0 \mathcal{L}^n(\Omega)} (\mathcal{E}(u(t)) - \mathcal{E}(u(t+\mu_0)))^{1/2}.
\end{aligned}$$

Then, since $\mathcal{E}$ is non-increasing and bounded below, the assertion follows by letting $t \to \infty$ in the previous inequality. Let now $q \in [1, p^*)$ and assume now by contradiction that along a diverging sequence of times $(t_j)$, we get

$$\sup_{\mu\in[0,\mu_0]} \|u(t_j) - u(t_j+\mu)\|_{L^q(\Omega)} \ge \sigma > 0,$$

for some positive constant $\sigma$ and all $j$ large. In particular, there is a sequence $\mu_j \subset [0, \mu_0]$ such that $\|u(t_j) - u(t_j + \mu_j)\|_{L^q(\Omega)} \ge \sigma > 0$ for all $j$ large. In light of (3.3), by Rellich compactness Theorem, up to a subsequence, it follows that $u(t_j) \to \xi_1$ in $L^q(\Omega)$ as $j \to \infty$ and $u(t_j + \mu_j) \to \xi_2$ in $L^q(\Omega)$ as $j \to \infty$, yielding $\|\xi_2 - \xi_1\|_{L^q(\Omega)} \ge \sigma > 0$. In particular $\xi_1 \ne \xi_2$. On the other hand, from (3.5) we immediately get $\|\xi_2 - \xi_1\|_{L^1} = 0$, leading to a contradiction. The second part of the statement has an analogous proof assuming by contradiction that there exists $\sigma > 0$ and a diverging sequence of times $(t_j)$ such that

$$\sup_{\mu\in[0,\mu_0]} \|u(t_j) - u(t_j+\mu)\|_{W_0^{1,p}(\Omega)} \ge \sigma > 0,$$

and then exploiting the relative compactness of $\{u(t) : t > 1\}$ in $W_0^{1,p}(\Omega)$. $\square$

On $W_0^{1,p}(\Omega)$ the functional $\mathcal{E}$ is defined by setting

$$\mathcal{E}(u) = \frac{1}{p} \int_\Omega a(u) |\nabla u|^p - \int_\Omega F(u). \tag{3.6}$$

and it is merely continuous, although its directional derivatives exist along smooth directions and

$$\mathcal{E}'(u)(\varphi) = \int_\Omega a(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \frac{1}{p} \int_\Omega a'(u) |\nabla u|^p \varphi - \int_\Omega f(u) \varphi.$$

We now recall an important compactness result (see e.g. [CD, Sq1]).

**Lemma 3.2.** *Let conditions (3.1) and (3.2) hold. Assume that $(u_h) \subset W_0^{1,p}(\Omega)$ is a bounded sequence and*

$$\langle w_h, \varphi\rangle = \int_\Omega a(u_h)|\nabla u_h|^{p-2}\nabla u_h \cdot \nabla\varphi + \frac{1}{p}\int_\Omega a'(u_h)|\nabla u_h|^p\varphi$$

*for every $\varphi \in C_c^\infty(\Omega)$, where $(w_h)$ is strongly convergent in $W^{-1,p'}(\Omega)$. Then $(u_h)$ admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.*

**Lemma 3.3.** *Let conditions (3.1) and (3.2) hold. Assume that there exist $C_1, C_2 > 0$ such that*

$$|f(s)| \leq C_1 + C_2|s|^r, \qquad \text{for all } s \in \mathbb{R}, \tag{3.7}$$

*for some $r \in [1, p^* - 1)$. Let $u : [0,\infty) \times \Omega \to \mathbb{R}$ be a global solution to problem (E), with $p > \frac{2n}{n+2}$. Then, for every diverging sequence $(\tau_j)$ there exists a diverging sequence $(t_j)$ with $t_j \in [\tau_j, \tau_j + 1]$ such that*

$$u(t_j) \to z \quad \text{in } W_0^{1,p}(\Omega) \text{ as } j \to \infty, \tag{3.8}$$

*where either $z = 0$ or $z$ is a solution to problem (S). In addition, it holds*

$$\lim_{t\to\infty} \sup_{\mu\in[0,\mu_0]} \|u(t_j + \mu) - z\|_{L^q(\Omega)} = 0, \quad \text{for all } q \in [1, p^*),$$

*for all fixed $\mu_0 > 0$.*

*Proof.* By the definition of solution, for all $\varphi \in C_c^\infty(\Omega)$ and for a.e. $t > 0$, we have

$$\begin{aligned}\int_\Omega u_t(t)\varphi dx + \int_\Omega a(u(t))|\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla\varphi dx \\ + \int_\Omega \frac{a'(u(t))}{p}|\nabla u(t)|^p\varphi dx = \int_\Omega f(u(t))\varphi dx.\end{aligned} \tag{3.9}$$

By means of the summability given by (3.4) it follows that, for every diverging sequence $(\tau_j) \subset \mathbb{R}^+$, there exists a diverging sequence $(t_j)$ with $t_j \in [\tau_j, \tau_j + 1]$, $j \geq 1$, such that

$$\Lambda_j = \int_\Omega |u_t(t_j)|^2 dx \to 0, \quad \text{as } j \to \infty. \tag{3.10}$$

Let us now define the sequence $(w_j)$ in $W^{-1,p'}(\Omega)$ by

$$\langle w_j, \varphi\rangle = \langle w_j^1, \varphi\rangle + \langle w_j^2, \varphi\rangle, \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega),$$

where we have set

$$\langle w_j^1, \varphi\rangle = \int_\Omega f(u(t_j))\varphi, \quad \langle w_j^2, \varphi\rangle = -\int_\Omega u_t(t_j)\varphi\, dx, \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

We recall that, under the growth condition (3.7), the map

$$W_0^{1,p}(\Omega) \ni u \mapsto f(u) \in W^{-1,p'}(\Omega)$$

is completely continuous, and hence, up to a further subsequence, we have

$$w_j^1 \to \mu, \qquad \text{in } W^{-1,p'}(\Omega) \text{ as } j \to \infty,$$

for some $\mu \in W^{-1,p'}(\Omega)$. Turning to the sequence $(w_j^2)$, notice that in view of (3.10), exploiting the fact that $p^* > 2$ since of the assumption $p > \frac{2n}{n+2}$, by Hölder inequality we get

$$\|w_j^2\|_{W^{-1,p'}(\Omega)} = \sup\left\{|\langle w_j, \varphi\rangle| : \varphi \in W_0^{1,p}(\Omega),\ \|\varphi\|_{W_0^{1,p}(\Omega)} \le 1\right\} \le C\Lambda_j,$$

for some positive constant $C$. Then $w_j^2 \to 0$ in $W^{-1,p'}(\Omega)$ as $j \to \infty$ and, in conclusion, $w_j \to \mu$ in $W^{-1,p'}(\Omega)$ as $j \to \infty$. Furthermore, by means of (3.9), we conclude that

$$\langle w_j, \varphi\rangle = \int_\Omega a(u(t_j))|\nabla u(t_j)|^{p-2}\nabla u(t_j)\cdot\nabla\varphi + \frac{1}{p}\int_\Omega a'(u(t_j))|\nabla u(t_j)|^p\varphi, \tag{3.11}$$

for all $\varphi \in C_c^\infty(\Omega)$. We have thus proved that $(u(t_j)) \subset W_0^{1,p}(\Omega)$ is in the framework of the compactness Lemma 3.2. In turn, by Lemma 3.2, up to a subsequence $(u(t_j))$ is strongly convergent to some $z$ in $W_0^{1,p}(\Omega)$, as $j \to \infty$. In particular, $u(t_j, x) \to z(x)$ and $\nabla u(t_j, x) \to \nabla z(x)$ for a.e. $x \in \Omega$, as $j \to \infty$. Since

$$|a'(u(t_j, x))|\nabla u(t_j, x)|^p\varphi(x)| \le C|\nabla u(t_j, x)|^p, \quad \text{for all } j \ge 1 \text{ and } x \in \Omega,$$

and $|\nabla u(t_j, x)|^p \to |\nabla z(x)|^p$ in $L^1(\Omega)$ as $j \to \infty$, we have

$$\lim_{j\to\infty}\int_\Omega a'(u(t_j))|\nabla u(t_j)|^p\varphi dx = \int_\Omega a'(z)|\nabla z|^p\varphi dx$$

by generalized Lebesgue dominated convergence theorem. Also, as

$$a(u(t_j, x))|\nabla u(t_j, x)|^{p-2}\nabla u(t_j, x) \to a(z(x))|\nabla z(x)|^{p-2}\nabla z(x),$$

and

$$a(u(t_j))|\nabla u(t_j)|^{p-2}\nabla u(t_j) \quad \text{is bounded in } L^{p'}(\Omega),$$

we have

$$\lim_{j\to\infty}\int_\Omega a(u(t_j))|\nabla u(t_j)|^{p-2}\nabla u(t_j)\cdot\nabla\varphi\, dx = \int_\Omega a(z)|\nabla z|^{p-2}\nabla z\cdot\nabla\varphi\, dx$$

Finally, since $f(u(t_j, x)) \to f(z(x))$ a.e. in $\Omega$, as $j \to \infty$, we get

$$\lim_{j\to\infty}\langle w_j, \varphi\rangle = \lim_{j\to\infty}\int_\Omega f(u(t_j))\varphi\, dx = \int_\Omega f(z)\varphi\, dx.$$

In particular, letting $j \to \infty$ in formula (3.11), it follows that $z$ is a (possibly zero) weak solution to problem

$$-\mathrm{div}(a(z)|\nabla z|^{p-2}\nabla z) + \frac{a'(z)}{p}|\nabla z|^p = f(z), \quad \text{in } \Omega.$$

The last assertion of the statement is just a combination of (3.8) with Lemma 3.1. □

**Lemma 3.4.** *Let $u_0 \in \mathcal{A}$ and let $u : [0, \infty) \times \Omega \to \mathbb{R}^+$ be the corresponding global solution to problem (E). Then the $\omega$-limit set $\omega(u_0)$ only contains positive (possibly identically zero) solutions of problem (S).*

*Proof.* Let $z \in \omega(u_0)$. Therefore, there exists a diverging sequence $(t_j) \subset \mathbb{R}^+$ such that $u(t_j)$ converges to $z$ in $W_0^{1,p}(\Omega)$, as $j \to \infty$. Let now $\varphi \in C_c^\infty(\Omega)$ be a given test function with $\|\varphi\|_{C^1} \leq 1$. Multiply problem $(E)$ by $\varphi$ and integrate it in space over $\Omega$ and in time over $[t_j, t_j + \sigma_j]$, where $\sigma_j \in [\sigma, 1]$ for a fixed $\sigma > 0$, yielding

$$\begin{aligned}(3.12)\qquad \int_{t_j}^{t_j+\sigma_j}\int_\Omega u_t\varphi dx &+ \int_{t_j}^{t_j+\sigma_j}\int_\Omega a(u)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx\\ &+\frac{1}{p}\int_{t_j}^{t_j+\sigma_j}\int_\Omega a'(u)|\nabla u|^p\varphi dx = \int_{t_j}^{t_j+\sigma_j}\int_\Omega f(u)\varphi dx,\end{aligned}$$

for any $j \geq 1$. Now, by virtue of Lemma 3.1, it follows that

$$\begin{aligned}\Big|\int_{t_j}^{t_j+\sigma_j}\int_\Omega u_t\varphi dx\Big| &= \Big|\int_\Omega (u(t_j+\sigma_j)-u(t_j))\varphi dx\Big|\\ &\leq \int_\Omega |u(t_j+\sigma_j)-u(t_j)||\varphi| dx\\ &\leq C\|u(t_j+\sigma_j)-u(t_j)\|_{L^1} = o(1), \quad \text{as } j\to\infty.\end{aligned}$$

In particular, recalling that $u \in C([0,\infty), W_0^{1,p}(\Omega,\mathbb{R}^+))$, by applying the mean value theorem, we find a new diverging sequence $(\xi_j) \subset \mathbb{R}^+$ with $\xi_j \in [t_j, t_j+\sigma_j]$ such that

$$\begin{aligned}(3.13)\qquad \int_\Omega a(u(\xi_j))|\nabla u(\xi_j)|^{p-2}\nabla u(\xi_j)\cdot\nabla\varphi dx &+ \frac{1}{p}\int_\Omega a'(u(\xi_j))|\nabla u(\xi_j)|^p\varphi dx\\ &= \int_\Omega f(u(\xi_j))\varphi dx + o(1), \quad \text{as } j\to\infty.\end{aligned}$$

In general, the choice of the sequence $(\xi_j)$ may depend upon the particular test function $\varphi$ that was fixed. On the other hand, taking into account the second part of the statement of Lemma 3.1, without loss of generality we may assume that $\xi_j$ is independent of $\varphi$. In fact, denoting by $(\xi_j^0)$ and $(\xi_j^\varphi)$ the sequences satisfying the property above and related to a reference test functions $\varphi_0$ and to an arbitrary test function $\varphi$ respectively, and writing,

$$(3.14)\qquad u(\xi_j^0) - u(\xi_j^\varphi) = \beta_j, \qquad \text{where } \beta_j \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } j\to\infty,$$

where $\beta_j$ is independent of $\varphi$, we get

$$\begin{aligned}&\Big|\int_\Omega a(u(\xi_j^0))|\nabla u(\xi_j^0)|^{p-2}\nabla u(\xi_j^0)\cdot\nabla\varphi dx - \int_\Omega a(u(\xi_j^\varphi))|\nabla u(\xi_j^\varphi)|^{p-2}\nabla u(\xi_j^\varphi)\cdot\nabla\varphi\, dx\Big|\\ &= \Big|\int_\Omega \big(a(u(\xi_j^0))|\nabla u(\xi_j^0)|^{p-2}\nabla u(\xi_j^0) - a(u(\xi_j^\varphi))|\nabla u(\xi_j^\varphi)|^{p-2}\nabla u(\xi_j^\varphi)\big)\cdot\nabla\varphi\, dx\Big|\\ &\leq \int_\Omega \big|a(u(\xi_j^0))|\nabla u(\xi_j^0)|^{p-2}\nabla u(\xi_j^0) - a(u(\xi_j^\varphi))|\nabla u(\xi_j^\varphi)|^{p-2}\nabla u(\xi_j^\varphi)\big| dx = \varpi_j\end{aligned}$$

where $\varpi_j \to 0$, as $j \to \infty$, by the generalized Lebesgue dominated convergence. In a similar fashion one can treat the other terms. By the relative compactness of the trajectory $u(t)$ into $W_0^{1,p}(\Omega)$, there exists a subsequence $(\xi_{j_k})$, that we rename into $(\xi_j)$, such that $u(\xi_j)$ is

strongly convergent to some $\hat{z}$ in $W_0^{1,p}(\Omega)$ as $j \to \infty$. Then, letting $j \to \infty$ in (3.13), the generalized Lebesgue dominated convergence yields

$$\int_\Omega a(\hat{z})|\nabla \hat{z}|^{p-2}\nabla \hat{z} \cdot \nabla \varphi dx + \frac{1}{p}\int_\Omega a'(\hat{z})|\nabla \hat{z}|^p \varphi dx = \int_\Omega f(\hat{z})\varphi dx, \quad \forall \varphi \in C_c^\infty(\Omega),$$

showing that $\hat{z}$ is a solution of problem $(S)^2$. Then, on one hand, we have $u(t_j) \to z$ in $W_0^{1,p}(\Omega)$ as $j \to \infty$ and, on the other hand, $u(\xi_j) \to \hat{z}$ in $W_0^{1,p}(\Omega)$ as $j \to \infty$. In light of the second part of the statement of Lemma 3.1, we have

$$\begin{aligned}\|z - \hat{z}\|_{W_0^{1,p}(\Omega)} &\le \|z - u(t_j)\|_{W_0^{1,p}(\Omega)} + \|u(t_j) - u(\xi_j)\|_{W_0^{1,p}(\Omega)} + \|u(\xi_j) - \hat{z}\|_{W_0^{1,p}(\Omega)} \\ &\le \sup_{\mu \in [0,1]} \|u(t_j) - u(t_j + \mu)\|_{W_0^{1,p}(\Omega)} + o(1) = o(1),\end{aligned}$$

as $j \to \infty$, yielding $\hat{z} = z$ and concluding the proof. □

**Remark 3.5.** *Forcing the nonlinearity $f$ to be zero for negative values, the sign condition on $a'$ usually induces global solutions starting from positive initial data to remain positive for all times $t > 0$. In fact, let us definite $\hat{f} : \mathbb{R} \to \mathbb{R}$ by setting*

$$\hat{f}(s) = \begin{cases} f(s) & \text{if } s \ge 0, \\ 0 & \text{if } s < 0, \end{cases} \tag{3.15}$$

*assume that $u_0 \ge 0$ a.e. in $\Omega$ and, furthermore, that*

$$a'(s) \le 0, \quad \text{for all } s \le 0. \tag{3.16}$$

*Then the solutions to the problem*

$$\begin{cases} u_t - \operatorname{div}(a(u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}a'(u)|\nabla u|^p = \hat{f}(u) & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{in } (0,\infty) \times \partial\Omega, \end{cases} \tag{3.17}$$

*satisfy $u(x,t) \ge 0$, for a.e. $x \in \Omega$ and all $t \ge 0$. In fact, let us consider the Lipschitz function $Q : \mathbb{R} \to \mathbb{R}$ being defined by*

$$Q(s) = \begin{cases} 0 & \text{if } s \ge 0, \\ s & \text{if } s \le 0. \end{cases}$$

*Testing equation (3.17) by $Q(u) \in W_0^{1,p}(\Omega)$ (which is an admissible test by (3.16) in view of the result of [BB] being $a'(u)|\nabla u|^p Q(u) \ge 0$ a.e. in $\mathbb{R}^n$) and recalling (3.15), we get*

$$\int_\Omega u_t Q(u) dx + \int_\Omega a(u)|\nabla u|^{p-2}\nabla u \nabla Q(u) dx + \frac{1}{p}\int_\Omega a'(u)|\nabla u|^p Q(u) dx = \int_\Omega \hat{f}(u) Q(u) dx.$$

*Notice that it holds*

$$\int_\Omega u_t Q(u) dx = \frac{1}{2}\frac{d}{dt}\int_\Omega Q^2(u) dx, \qquad \int_\Omega \hat{f}(u) Q(u) dx = 0.$$

[2] Notice that we assumed $\|\varphi\|_{C^1} \le 1$. It is easily seen, anyway, that this assumption may be dropped via rescaling.

*as well as*

$$\int_\Omega a(u)|\nabla u|^{p-2}\nabla u\cdot\nabla Q(u)dx = \int_{\Omega\cap\{u\le 0\}} a(u)|\nabla u|^p dx \ge 0,$$
$$\int_\Omega a'(u)|\nabla u|^p Q(u)dx = \int_{\Omega\cap\{u\le 0\}} a'(u)u|\nabla u|^p dx \ge 0.$$

*In turn we conclude that*

$$\frac{d}{dt}\int_\Omega Q^2(u(t))dx \le 0,$$

*which yields the assertion by the definition of $Q$ and the assumption that the initial datum $u_0$ is positive, being $Q(u(t)) = 0$, for all times $t > 0$.*

## 4. Proof of the results

Finally we can prove the main results.

**Proof of Theorem 1.2.** Assume that $f$ is strictly positive in $(0,\infty)$ and $\Omega$ is strictly convex with respect to a direction, say $x_1$, and symmetric with respect to the hyperplane $\{x_1 = 0\}$. By Proposition 2.5, since $\Lambda = 0$ in this case, it follows $u(x_1, x') \le u(-x_1, x')$ for $x_1 \le 0$. In the same way one can prove that $u(x_1, x') \ge u(-x_1, x')$. Therefore

$$u(x_1, x') = u(-x_1, x'),$$

that is $u$ belongs to the class $\mathcal{S}_{x_1}$, since the monotonicity follows by (2.37) in Proposition 2.5. Finally, if $\Omega$ is a ball, by repeating this argument along any direction, it follows that $u$ belongs to $\mathcal{R}$.

**Proof of Theorem 1.4.** Part (a) of the assertion follows by combining Theorem 1.2 with Lemma 3.3. According to the notations in the statement of Theorem 1.4, if $z \ne 0$ and $z \in W_0^{1,p} \cap L^\infty(\Omega)$ then by the regularity results of [Di, Lie, Tol] it follows that $z \in C^{1,\alpha}(\bar{\Omega})$ and hence the assumptions of Theorem 1.2 are fulfilled. Part (b) follows by combining Theorem 1.2 with a uniqueness result (of radial solutions) due to Erbe-Tang [ET, Main Theorem, p.355].

**Proof of Theorem 1.7.** Part (a) of the assertion follows from a combination of Theorem 1.2 with Lemma 3.4, while part (b) follows by combining Theorem 1.2 with a uniqueness result (of radial solutions) due to Erbe-Tang [ET, Main Theorem, p.355].

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